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ABBE'S TEST OF APLANATISM, AND A SIMPLE APERTOMETER DERIVED THEREFROM.

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(*Read June* 19*th*, 1903.)

The Abbe-Helmholtz sine-law expresses, as is well known, the necessary and sufficient condition for the production, by the different zones of a wide-angle optical system, of equal-sized images of an indefinitely small object on the axis of the system, and in a plane at right angles to that axis.

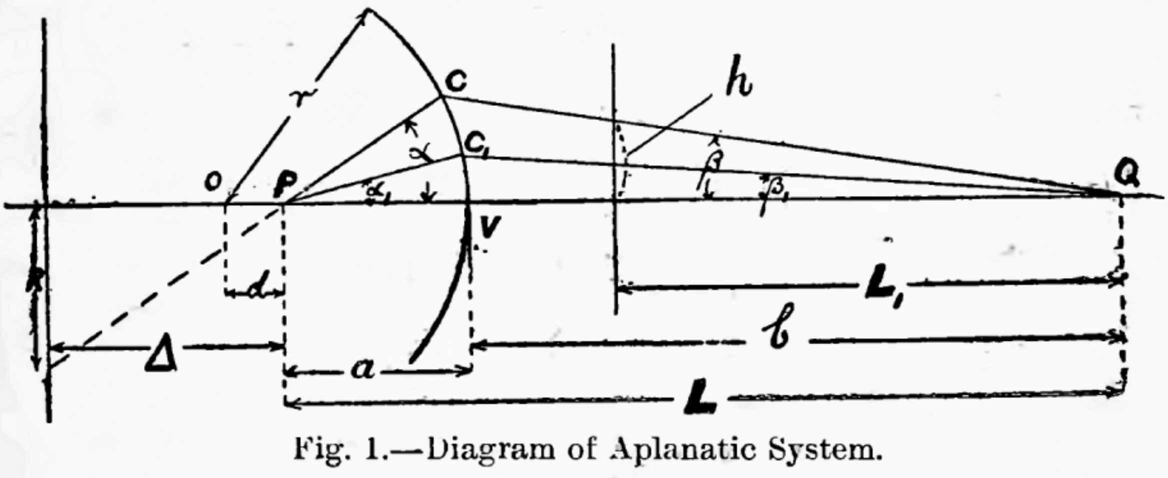


Fig. 1.—Diagram of Aplanatic System.

Let P and Q be a pair of conjugate and aplanatic foci, on the axis of a wide-angle optical system, then the sine-law states that the sine of the angle α, which any ray makes with the axis in passing from the point P, must bear a constant ratio to the sine of the angle β, which the corresponding conjugate ray makes with the axis, when passing through the point Q. If the points P and Q be immersed in media with refractive indices

μ and μ_1, respectively, and if M equal the magnification produced by the system, the sine-law fully stated takes the form—

$$\frac{\sin \alpha}{\sin \beta} = M \frac{\mu_1}{\mu} = K \text{ (a constant)}^* \quad . \quad . \quad . \quad . \quad (1)$$

In general, a ray, passing from the point P to the point Q, and undergoing a total deviation equal to the sum of the angles α and β, would suffer, in any practical optical system, many refractions, which, however, it is not necessary for our purpose to consider. All that we are concerned with is the total deviation, and this may be looked upon as though produced by a single refraction only, at the point C, obtained by producing the incident ray and its conjugate ray until they meet as shown. To the point C the name *chief* point has been given by Professor S. P. Thompson.

In a similar way, let C_1 be the point of intersection of a second pair of rays, making angles α_1 and β_1, respectively, with the axis. Then from simple geometry we have—

$$\frac{\sin \alpha}{\sin \beta} = \frac{CQ}{CP},$$

and—

$$\frac{\sin \alpha_1}{\sin \beta_1} = \frac{C_1 Q}{C_1 P};$$

and, since the ratio of the sines is constant,—

$$\frac{CQ}{CP} = \frac{C_1 Q}{C_1 P}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and so for any pair of rays. It follows from the constancy of this ratio and the fixed distance PQ, that the chief points must lie upon a curve, which is the locus of a triangle constructed on a given base and with a constant ratio between the lengths of its other two sides. This locus is a circle,† with its centre on the axis PQ, and cutting it, say, at V. Let $PV = a$, and $VQ = b$; then r, the radius of this circle, is obtained from—

$$r = \frac{a \,.\, b}{b - a}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and, if a be less than b, the centre O of this circle is at a point on the axis such that we have for d the distance PO:

$$d = \frac{a^2}{b - a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We have thus arrived at the following important result:—*In*

* See *Heath's Geometrical Optics*, 1887, p. 255.

† Briggs and Bryan, *Co-ordinate Geometry*, 3rd edition, Part I., p. 186.

*any wide-angle optical system, which satisfies the sine-condition for a pair of conjugate foci, the equivalent refracting surface for these foci is a part of a sphere.**

In the case of the microscope objective, with which we are principally concerned, the image is always formed in air, hence $\mu_1 = 1$ in equation 1, and for a pair of conjugate rays meeting in the vertex V,—

$$\frac{b}{a} = \frac{\sin \alpha}{\sin \beta} = \frac{\mu}{M} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Putting L for $a + b$, the distance PQ, and M/μ for b/a, we can write equation 3 in the form—

$$r = \frac{\mu L M}{M^2 - \mu^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and equation 4 as—

$$d = \frac{\mu^2 L}{M^2 - \mu^2} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

An example will show the use and application of the last two equations. A dry lens, of a focal length of 15·8 mm., gave in a plane 205 mm. above the plane of the object, on the stage of the microscope, a magnification of 11·5. Substituting these values in equation 6, and remembering that $\mu = 1$ in this case, we have, for the value of the radius of the equivalent refracting spherical surface—

$$r = \frac{205 \times 11{\cdot}5}{(11{\cdot}5)^2 - 1} = 18 \text{ mm.}$$

And obviously, so long as $a < b$, this surface must be convex on its upper surface. By substituting in equation 7, we get for the distance d of the centre of curvature O below the aplanatic focus P,—

$$d = \frac{205}{(11{\cdot}5)^2 - 1} = 1{\cdot}6 \text{ mm.};$$

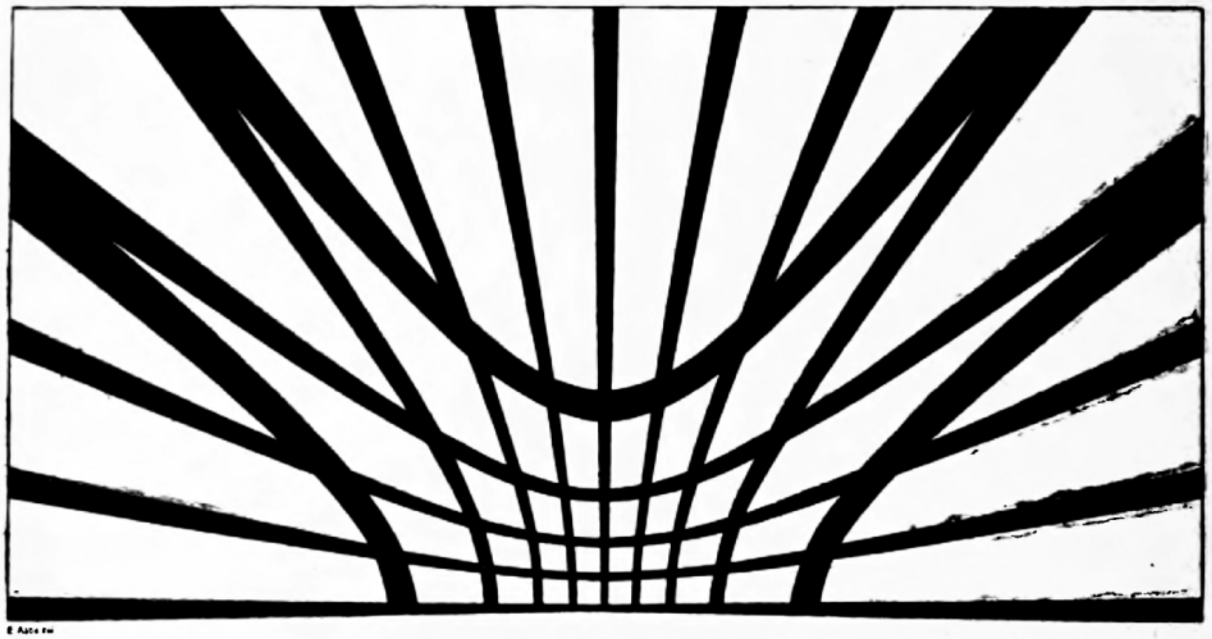
and again, so long as $a < b$, O is below P. Thus, in a very simple and practical way, it is possible to determine for any aplanatic system, from the distance between the aplanatic foci

* This proposition is well known for the particular case in which one of the aplanatic foci is at infinity, as for a telescope object-glass; but, so far as I can discover, the general proof given above, simple though it is, and important as it appears to be, does not occur in any English book on the subject. Dr. von Rohr, of Jena, has, however, since the reading of the paper, drawn my attention to an article by Mittenzwei in the *Jahrbuch für Photographie*, 1888, pp. 317-20, which clearly anticipates my proposition.

and the magnification, the radius of curvature of the equivalent refracting surface and the point at which the latter cuts the axis.

Abbe's Test of Aplanatism.—In the year 1879,* Abbe, wishing to ascertain to what extent objectives, made before the formulation of the sine-law, satisfied that law, invented the test diagram shown by Fig. 2. The problem was to find the nature of the curves, which, drawn upon a flat surface placed normal to the axis of a microscope and at a given distance below the lower focus of the objective to be tested, should project into the upper focal plane of the objective as a rectangular network of equi-thick

Fig. 2.



ABBE'S TEST FOR APLANATISM
($\Delta = 12{\cdot}5$ mm.)

and equi-distant parallel straight lines, in the event of the sine-law being fulfilled. These curves, by a method to be subsequently described, can be shown to be hyperbolas. To use the diagram, it should be placed upon the stage of the microscope, and the object to be tested focussed upon the middle point of the bottom line. The body of the microscope, carrying the objective with it, should then be racked back through 12·5 mm. Upon removing the eye-piece and looking down the tube, one-half of the back of the objective will be found to be occupied by an image of the diagram, in the form of the net-work referred to, if the objective is a good one.

* See *Gesammelte Abhandlungen von Ernst Abbe*, p. 226.

It occurred to the author of this paper that Abbe's test might be modified to project into the upper focal plane of an aplanatic objective to be tested for numerical aperture—*not* for aplanatism—

Fig. 3.

CHESHIRE'S APERTOMETER.

($\Delta = 25$ mm.)

a series of equi-thick equi-distant concentric circles, each of which should correspond to a definite and predetermined N.A. The result is shown by Fig. 3.

Theory of the New Apertometer.—Returning to Fig. 1, let us consider the ray PCQ intersecting the second (back) principal focal plane of the objective, at a distance h from the axis; and let the distance of this plane from the point Q $= L_1$. Then, Q being in air, we have from equation 5,—

$$\sin \beta = \frac{\mu \sin \alpha}{M} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Since the angle β, in a microscope system, never exceeds a few degrees, its tangent may be taken as equal to its sine; hence—

$$\sin \beta = \frac{h}{L_1}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and the magnification at Q is equal to the distance L_1, divided by the back focal length of the objective system; or—

$$M = \frac{L_1}{f} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Combining equations 9 and 10 we obtain—

$$\sin \beta = \frac{h}{Mf};$$

and substituting in equation 8—

$$\frac{h}{f} = \mu \sin \alpha = \text{N.A.}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

a well-known result which tells us that for objectives of a given focal length their N.A.'s vary directly as the effective diameters in the upper focal planes. Imagine now the point P in air ($\mu = 1$), and the ray CP produced backwards until it intersects, at a distance R from the axis, a plane normal to the axis, and at a distance $\triangle$ from the aplanatic focus P; and further, let us suppose that rays can only enter the system through a very small stop at P. Then to find the radius R of a circle which, placed normal to the axis and at a distance $\triangle$ from the aplanatic point P, shall project so as to completely fill the effective opening in the upper focal plane of an objective with a given N.A., we have only to remember that the N.A. $= \sin \alpha$, and that $R/\triangle = \tan \alpha$, to obtain the desired equation,—

$$R = \triangle \cdot \tan (\sin^{-1} \text{N.A.}) \quad . \quad . \quad . \quad . \quad . \quad (12)$$

A circle drawn with such a radius, and placed at the distance $\triangle$, will fill any objective with the given N.A., no matter what its focal length may be.

Agreeing, then, upon some convenient value for $\triangle$, it is a very simple matter to calculate the various values of R for a series of circles which shall correspond, in the way described, to N.A.'s of 0·1, 0·2, 0·3, and so on. These circles would, as is obvious from a consideration of equation 11, project into the upper focal plane of any objective as a number of equi-distant, concentric circles, but they would not, in general, be of equal thickness. To secure this object, it is necessary to calculate, for each circle on the diagram, a thickness which corresponds to an equal increment of the N.A. Thus, instead of calculating the radius of a circle to project as equivalent to a N.A. of 0·5, say, it is better to calculate for 0·49 and 0·51, draw the two circles, and blacken the space between them. The difference between the N.A. represented by the circle of the inner edge of any line and that represented by the circle of the outer edge is thus in every case equal to 0·02 N.A. The following table has been calculated in this way for a value of $\triangle = 1$, and for N.A.'s commencing at 0·1 and proceeding by steps of 0·1 to 0·9.

	R.	N.A.
1st Circle	$\begin{cases}0·09\\0·11\end{cases}$	0·09 0·11
2nd "	$\begin{cases}0·19\\0·22\end{cases}$	0·19 0·21
3rd "	$\begin{cases}0·30\\0·33\end{cases}$	0·29 0·31
4th "	$\begin{cases}0·42\\0·45\end{cases}$	0·39 0·41
5th "	$\begin{cases}0·56\\0·59\end{cases}$	0·49 0·51
6th "	$\begin{cases}0·73\\0·77\end{cases}$	0·59 0·61
7th "	$\begin{cases}0·96\\1·01\end{cases}$	0·69 0·71
8th "	$\begin{cases}1·29\\1·38\end{cases}$	0·79 0·81
9th "	$\begin{cases}1·95\\2·19\end{cases}$	0·89 0·91

To use this table for the calculation of the radii of the N.A. circles for any other value of $\triangle$, it is only necessary to remember that R must be read in the unit selected for $\triangle$, and must be multiplied by it. Thus if $\triangle$ be taken as 2 inches, each number under R must be multiplied by 2, to obtain the desired radii in inches. Similarly, if $\triangle$ be taken in centimetres, R must be read as centimetres. Should the apertometer, when made, be too large to be accommodated on the stage of the microscope, with its centre in the axis of the instrument, it should be cut down on one side until it is possible to do so.

In using this apertometer it is necessary to observe the image in the upper focal plane of the objective, either directly, or after it has been magnified in some way. Whatever method is adopted it is important that a small stop should be used, placed virtually at the point P, to sharply define the apex of the cone of light taken up by the objective. One of the following methods may be employed:—

1. The unassisted eye may be used, in which case the image of the eye-pupil formed by the objective serves to define the point P. Care should be taken to keep the eye fixed during the taking of a reading.

2. The observation may be made through a 2—3 mm. hole in a plate on the top of the draw-tube, replacing the ordinary eye-piece.

3. The bottom of the draw-tube may be fitted with a low-power lens, with a small stop near its upper focal plane—this lens forming with an eye-piece a low-power telescope.

4. By using a low-power eye-piece—the lower the better—fitted with a 2—3 mm. stop in the usual place, to form in the eye-ring an image of the image in the upper focal plane of the objective. The eye-ring may then be examined with a hand-magnifier.

The 2nd and 4th methods will generally be found the most convenient—the first for testing low-power objectives, and the second for testing high-power ones.

[The diagrams on the loose plate accompanying this paper are intended to be cut out and used on the microscope in the way described. The apertometer disc should be placed upon the microscope stage, with its centre in the axis of the instrument, upon which the objective to be tested should be focussed. Then, upon racking back the body through 25 mm., and removing the eye-piece, the N.A. of the objective will be found projected in its upper focal plane.]

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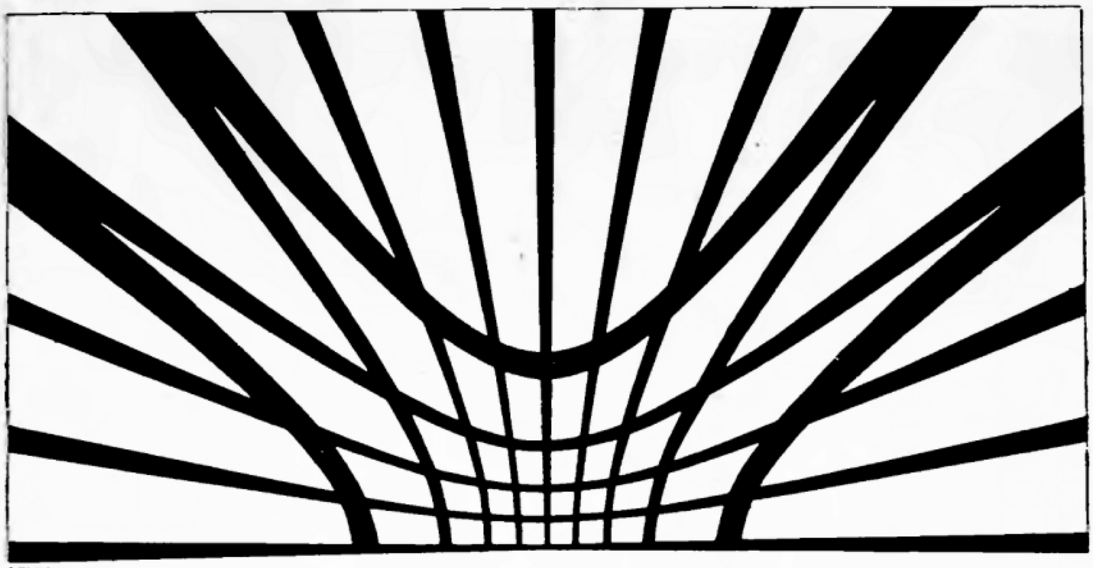

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